

THE SUPERMAGIC SQUARE IN CHARACTERISTIC 3 AND JORDAN SUPERALGEBRAS

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Dedicated to Ivan Shestakov, on the occasion of his 60th birthday

ABSTRACT. Recently, the classical Freudenthal Magic Square has been extended over fields of characteristic 3 with two more rows and columns filled with (mostly simple) Lie superalgebras specific of this characteristic. This *Supermagic Square* will be reviewed and some of the simple Lie superalgebras that appear will be shown to be isomorphic to the Tits-Kantor-Koecher Lie superalgebras of some Jordan superalgebras.

INTRODUCTION

The classical Freudenthal Magic Square, which contains in characteristic 0 the exceptional simple finite dimensional Lie algebras, other than G_2 , is usually constructed based on two ingredients: a unital composition algebra and a central simple degree 3 Jordan algebra (see [Sch95, Chapter IV]). This construction, due to Tits, does not work in characteristic 3.

A more symmetric construction, based on two unital composition algebras, which play symmetric roles, and their triality Lie algebras, has been given recently by several authors ([AF93], [BS], [LM02] [LM04]). Among other things, this construction has the advantage of being valid too in characteristic 3. Simpler formulas for triality appear if symmetric composition algebras are used, instead of the more classical unital composition algebras ([Eld04, Eld07a]).

But the characteristic 3 presents an exceptional feature, as only over fields of this characteristic there are nontrivial composition superalgebras, which appear in dimensions 3 and 6. The unital such composition superalgebras were discovered by Shestakov [She97]. This fact allows to extend Freudenthal Magic Square ([CE07a]) with the addition of two further rows and columns, filled with (mostly simple) Lie superalgebras, specific of characteristic 3, which had appeared first (with one exception) in [Eld06] and [Eld07b].

Most of the Lie superalgebras in characteristic 3 that appear in the Supermagic Square have been shown to be related to degree three simple Jordan algebras in [CE07b].

The aim of this paper is to show that some of the Lie superalgebras in the Supermagic Square are isomorphic to the Tits-Kantor-Koecher Lie superalgebras of some distinguished Jordan superalgebras.

More specifically, let S_i denote the split para-Hurwitz algebra of dimension $i = 1, 2$ or 4, and let S be the para-Hurwitz superalgebra associated to the unital composition superalgebra C (see Section 1 for definitions and notations). Let $\mathfrak{g}(S_i, S)$ be the corresponding entry in the Supermagic Square. Then $\mathfrak{g}(S_1, S)$ was shown in [CE07b] to be isomorphic to the Lie superalgebra of derivations of the

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Jordan superalgebra $J = H_3(C)$ of hermitian 3×3 matrices over C . Here the following results will be proved:

- (i) The Lie superalgebras $\mathfrak{g}(S_2, S)$ in the second row of the Supermagic Square will be shown to be isomorphic to the projective structure superalgebras of the Jordan superalgebras $J = H_3(C)$. Here the structure superalgebra is $\mathfrak{str}(J) = L_J \oplus \mathfrak{der}(J)$ and the projective structure superalgebra $\mathfrak{pstr}(J)$ is the quotient of $\mathfrak{str}(J)$ by its center. (See Theorem 3.3 and Corollary 3.5.)
- (ii) The Lie superalgebras $\mathfrak{g}(S_4, S)$ in the third row of the Supermagic Square will be shown to be isomorphic to the Tits-Kantor-Koecher Lie superalgebras of the Jordan superalgebras $J = H_3(C)$. (See Theorem 3.8 and Corollary 3.10.)
- (iii) The Lie superalgebra $\mathfrak{g}(S_{1,2}, S_{1,2})$ will be shown to be isomorphic to the Tits-Kantor-Koecher Lie superalgebra of the nine dimensional Kac Jordan superalgebra K_9 . Note that the ten dimensional Kac Jordan superalgebra K_{10} is no longer simple in characteristic 3, but contains a nine dimensional simple ideal, which is K_9 . (See Theorem 4.2.)
- (iv) The Lie superalgebra $\mathfrak{g}(S_1, S_{1,2})$ will be shown to be isomorphic to the Tits-Kantor-Koecher Lie superalgebra of the three dimensional Kaplansky superalgebra K_3 . (See Corollary 4.3.)

The paper is structured as follows. In Section 1 the construction of the Supermagic Square in terms of two symmetric composition superalgebras will be reviewed. Then the relationship of the Lie superalgebras in the first row of the Supermagic Square with the Lie superalgebras of derivations of the Jordan superalgebras $J = H_3(C)$ above, proven in [CE07b], will be reviewed in Section 2. Section 3 will be devoted to the Lie superalgebras in the second and third rows of the Supermagic Square, while Section 4 will deal with the Lie superalgebra $\mathfrak{g}(S_{1,2}, S_{1,2})$ and the nine dimensional Kac Jordan superalgebra K_9 . It was Shestakov [She96] who first noticed that K_9 is isomorphic to the tensor product (in the graded sense) of two copies of the three dimensional Kaplansky Jordan superalgebra K_3 (this was further developed in [BE02]), and this is the key for the results in Section 4.

Unless otherwise stated, all the vector spaces and superspaces considered will be assumed to be finite dimensional over a ground field k of characteristic $\neq 2$.

1. THE SUPERMAGIC SQUARE

A quadratic superform on a \mathbb{Z}_2 -graded vector space $U = U_{\bar{0}} \oplus U_{\bar{1}}$ over a field k is a pair $q = (q_{\bar{0}}, b)$ where $q_{\bar{0}} : U_{\bar{0}} \rightarrow k$ is a quadratic form, and $b : U \times U \rightarrow k$ is a supersymmetric even bilinear form such that $b|_{U_{\bar{0}} \times U_{\bar{0}}}$ is the polar form of $q_{\bar{0}}$:

$$b(x_{\bar{0}}, y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}} + y_{\bar{0}}) - q_{\bar{0}}(x_{\bar{0}}) - q_{\bar{0}}(y_{\bar{0}})$$

for any $x_{\bar{0}}, y_{\bar{0}} \in U_{\bar{0}}$.

The quadratic superform $q = (q_{\bar{0}}, b)$ is said to be *regular* if the bilinear form b is nondegenerate.

Then a superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$ over k , endowed with a regular quadratic superform $q = (q_{\bar{0}}, b)$, called the *norm*, is said to be a *composition superalgebra* (see [EO02]) in case

$$q_{\bar{0}}(x_{\bar{0}}y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}})q_{\bar{0}}(y_{\bar{0}}), \quad (1.1a)$$

$$b(x_{\bar{0}}y, x_{\bar{0}}z) = q_{\bar{0}}(x_{\bar{0}})b(y, z) = b(yx_{\bar{0}}, zx_{\bar{0}}), \quad (1.1b)$$

$$b(xy, zt) + (-1)^{xy+yz+yz}b(zy, xt) = (-1)^{yz}b(x, z)b(y, t), \quad (1.1c)$$

for any $x_{\bar{0}}, y_{\bar{0}} \in C_{\bar{0}}$ and homogeneous elements $x, y, z, t \in C$. (As we are working in characteristic $\neq 2$, it is enough to consider equation (1.1c).)

As usual, the expression $(-1)^{yz}$ equals -1 if the homogeneous elements y and z are both odd, otherwise, it equals 1 .

The unital composition superalgebras are termed *Hurwitz superalgebras*, while a composition superalgebra is said to be *symmetric* in case its bilinear form is associative, that is,

$$b(xy, z) = b(x, yz),$$

for any x, y, z .

Hurwitz algebras are the well-known algebras that generalize the classical real division algebras of the real and complex numbers, quaternions and octonions. Over any algebraically closed field k , there are exactly four of them: k , $k \times k$, $\text{Mat}_2(k)$ and $C(k)$ (the split Cayley algebra), with dimensions $1, 2, 4$ and 8 .

Only over fields of characteristic 3 there appear nontrivial Hurwitz superalgebras (see [EO02]):

- Let V be a two dimensional vector space over a field k , endowed with a nonzero alternating bilinear form $\langle \cdot | \cdot \rangle$ (that is $\langle v | v \rangle = 0$ for any $v \in V$). Consider the superspace $B(1, 2)$ (see [She97]) with

$$B(1, 2)_{\bar{0}} = k1, \quad \text{and} \quad B(1, 2)_{\bar{1}} = V, \quad (1.2)$$

endowed with the supercommutative multiplication given by

$$1x = x1 = x \quad \text{and} \quad uv = \langle u | v \rangle 1$$

for any $x \in B(1, 2)$ and $u, v \in V$, and with the quadratic superform $q = (q_{\bar{0}}, b)$ given by:

$$q_{\bar{0}}(1) = 1, \quad b(u, v) = \langle u | v \rangle, \quad (1.3)$$

for any $u, v \in V$. If the characteristic of k is equal to 3 , then $B(1, 2)$ is a Hurwitz superalgebra ([EO02, Proposition 2.7]).

- Moreover, with V as before, let $f \mapsto \bar{f}$ be the associated symplectic involution on $\text{End}_k(V)$ (so $\langle f(u) | v \rangle = \langle u | \bar{f}(v) \rangle$ for any $u, v \in V$ and $f \in \text{End}_k(V)$). Consider the superspace $B(4, 2)$ (see [She97]) with

$$B(4, 2)_{\bar{0}} = \text{End}_k(V), \quad \text{and} \quad B(4, 2)_{\bar{1}} = V, \quad (1.4)$$

with multiplication given by the usual one (composition of maps) in $\text{End}_k(V)$, and by

$$v \cdot f = f(v) = \bar{f} \cdot v \in V,$$

$$u \cdot v = \langle \cdot | u \rangle v \in \text{End}_k(V)$$

for any $f \in \text{End}_k(V)$ and $u, v \in V$, where $\langle \cdot | u \rangle v$ denotes the endomorphism $w \mapsto \langle w | u \rangle v$; and with quadratic superform $q = (q_{\bar{0}}, b)$ such that

$$q_{\bar{0}}(f) = \det(f), \quad b(u, v) = \langle u | v \rangle,$$

for any $f \in \text{End}_k(V)$ and $u, v \in V$. If the characteristic is equal to 3 , $B(4, 2)$ is a Hurwitz superalgebra ([EO02, Proposition 2.7]).

Given any Hurwitz superalgebra C with norm $q = (q_{\bar{0}}, b)$, its standard involution is given by

$$x \mapsto \bar{x} = b(x, 1)1 - x.$$

A new product can be defined on C by means of

$$x \bullet y = \bar{x}\bar{y}. \quad (1.5)$$

The resulting superalgebra, denoted by \bar{C} , is called the *para-Hurwitz superalgebra* attached to C , and it turns out to be a symmetric composition superalgebra.

Given a symmetric composition superalgebra S , its *triality Lie superalgebra* $\mathbf{tri}(S) = \mathbf{tri}(S)_{\bar{0}} \oplus \mathbf{tri}(S)_{\bar{1}}$ is defined by:

$$\mathbf{tri}(S)_{\bar{i}} = \{(d_0, d_1, d_2) \in \mathfrak{osp}(S, q)_i^3 : \\ d_0(x \bullet y) = d_1(x) \bullet y + (-1)^{ix} x \bullet d_2(y) \ \forall x, y \in S_{\bar{0}} \cup S_{\bar{1}}\},$$

where $\bar{i} = \bar{0}, \bar{1}$, and $\mathfrak{osp}(S, q)$ denotes the associated orthosymplectic Lie superalgebra. The bracket in $\mathbf{tri}(S)$ is given componentwise.

Now, given two symmetric composition superalgebras S and S' , one can form (see [CE07a, §3], or [Eld04] for the non-super situation) the Lie superalgebra:

$$\mathfrak{g} = \mathfrak{g}(S, S') = (\mathbf{tri}(S) \oplus \mathbf{tri}(S')) \oplus (\oplus_{i=0}^2 \iota_i(S \otimes S')), \quad (1.6)$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$ ($i = 0, 1, 2$), with bracket given by:

- the Lie bracket in $\mathbf{tri}(S) \oplus \mathbf{tri}(S')$, which thus becomes a Lie subsuperalgebra of \mathfrak{g} ,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x')$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = (-1)^{d'_i x} \iota_i(x \otimes d'_i(x'))$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = (-1)^{x'y} \iota_{i+2}((x \bullet y) \otimes (x' \bullet y'))$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = (-1)^{xx' + xy' + yy'} b'(x', y') \theta^i(t_{x,y}) + (-1)^{yx'} b(x, y) \theta^i(t'_{x',y'})$,

for any $i = 0, 1, 2$ and homogeneous $x, y \in S$, $x', y' \in S'$, $(d_0, d_1, d_2) \in \mathbf{tri}(S)$, and $(d'_0, d'_1, d'_2) \in \mathbf{tri}(S')$. Here θ denotes the natural automorphism $\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$ in $\mathbf{tri}(S)$, while $t_{x,y}$ is defined by

$$t_{x,y} = (\sigma_{x,y}, \frac{1}{2}b(x,y)1 - r_x l_y, \frac{1}{2}b(x,y)1 - l_x r_y) \quad (1.7)$$

with $l_x(y) = x \bullet y$, $r_x(y) = (-1)^{xy} y \bullet x$, and

$$\sigma_{x,y}(z) = (-1)^{yz} b(x, z) y - (-1)^{x(y+z)} b(y, z) x \quad (1.8)$$

for homogeneous $x, y, z \in S$. Also θ' and $t'_{x',y'}$ denote the analogous elements for $\mathbf{tri}(S')$.

Over a field k of characteristic 3, let S_r ($r = 1, 2, 4$ or 8) denote the para-Hurwitz algebra attached to the split Hurwitz algebra of dimension r (this latter algebra being either k , $k \times k$, $\text{Mat}_2(k)$ or $C(k)$). Also, denote by $S_{1,2}$ the para-Hurwitz superalgebra $\overline{B(1,2)}$, and by $S_{4,2}$ the para-Hurwitz superalgebra $\overline{B(4,2)}$. Then the Lie superalgebras $\mathfrak{g}(S, S')$, where S, S' run over $\{S_1, S_2, S_4, S_8, S_{1,2}, S_{4,2}\}$, appear in Table 1, which has been obtained in [CE07a].

Since the construction of $\mathfrak{g}(S, S')$ is symmetric, only the entries above the diagonal are needed. In Table 1, $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ denote the simple exceptional classical Lie algebras, $\tilde{\mathfrak{e}}_6$ denotes a 78 dimensional Lie algebras whose derived Lie algebra is the 77 dimensional simple Lie algebra \mathfrak{e}_6 in characteristic 3. The even and odd parts of the nontrivial superalgebras in the table which have no counterpart in the classification in characteristic 0 ([Kac77a]) are displayed, *spin* denotes the spin module for the corresponding orthogonal Lie algebra, while (n) denotes a module of dimension n , whose precise description is given in [CE07a]. Thus, for example, $\mathfrak{g}(S_4, S_{1,2})$ is a Lie superalgebra whose even part is (isomorphic to) the direct sum of the symplectic Lie algebra \mathfrak{sp}_6 and of \mathfrak{sl}_2 , while its odd part is the tensor product of a 13 dimensional module for \mathfrak{sp}_6 and the natural 2 dimensional module for \mathfrak{sl}_2 .

A precise description of these modules and of the Lie superalgebras as Lie superalgebras with a Cartan matrix is given in [CE07a]. All the inequivalent Cartan matrices for these simple Lie superalgebras are listed in [BGL].

	S_1	S_2	S_4	S_8	$S_{1,2}$	$S_{4,2}$
S_1	\mathfrak{sl}_2	\mathfrak{pgl}_3	\mathfrak{sp}_6	\mathfrak{f}_4	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
S_2		$\mathfrak{pgl}_3 \oplus \mathfrak{pgl}_3$	\mathfrak{pgl}_6	$\tilde{\mathfrak{e}}_6$	$(\mathfrak{pgl}_3 \oplus \mathfrak{sl}_2) \oplus (\mathfrak{psl}_3 \otimes (2))$	$\mathfrak{pgl}_6 \oplus (20)$
S_4			\mathfrak{so}_{12}	\mathfrak{e}_7	$(\mathfrak{sp}_6 \oplus \mathfrak{sl}_2) \oplus ((13) \otimes (2))$	$\mathfrak{so}_{12} \oplus \mathit{spin}_{12}$
S_8				\mathfrak{e}_8	$(\mathfrak{f}_4 \oplus \mathfrak{sl}_2) \oplus ((25) \otimes (2))$	$\mathfrak{e}_7 \oplus (56)$
$S_{1,2}$					$\mathfrak{so}_7 \oplus 2\mathit{spin}_7$	$\mathfrak{sp}_8 \oplus (40)$
$S_{4,2}$						$\mathfrak{so}_{13} \oplus \mathit{spin}_{13}$

TABLE 1. Supermagic Square (characteristic 3)

With the exception of $\mathfrak{g}(S_{1,2}, S_{4,2})$, all these superalgebras have appeared previously in [Eld06] and [Eld07b].

2. JORDAN SUPERALGEBRAS

Given any Hurwitz superalgebra C over our ground field k , with norm $q = (q_0, b)$ and standard involution $x \mapsto \bar{x}$, the superalgebra $H_3(C)$ of 3×3 hermitian matrices over C , under the superinvolution given by $(a_{ij})^* = (\bar{a}_{ji})$, is a Jordan superalgebra under the symmetrized product

$$x \circ y = \frac{1}{2}(xy + (-1)^{xy}yx). \quad (2.1)$$

Let us consider the associated para-Hurwitz superalgebra $S = \bar{C}$, with multiplication $a \bullet b = \bar{a}\bar{b}$ for any $a, b \in C$. Then,

$$J = H_3(C) = \left\{ \begin{pmatrix} \alpha_0 & \bar{a}_2 & a_1 \\ a_2 & \alpha_1 & \bar{a}_0 \\ \bar{a}_1 & a_0 & \alpha_2 \end{pmatrix} : \alpha_0, \alpha_1, \alpha_2 \in k, a_0, a_1, a_2 \in S \right\} \quad (2.2)$$

$$= (\oplus_{i=0}^2 ke_i) \oplus (\oplus_{i=0}^2 \iota_i(S)),$$

where

$$e_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.3)$$

$$\iota_0(a) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_1(a) = 2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_2(a) = 2 \begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for any $a \in S$. Identify $ke_0 \oplus ke_1 \oplus ke_2$ to k^3 by means of $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 \simeq (\alpha_0, \alpha_1, \alpha_2)$. Then the supercommutative multiplication (2.1) becomes:

$$\begin{cases} (\alpha_0, \alpha_1, \alpha_2) \circ (\beta_1, \beta_2, \beta_3) = (\alpha_0 \beta_0, \alpha_1 \beta_1, \alpha_2 \beta_2), \\ (\alpha_0, \alpha_1, \alpha_2) \circ \iota_i(a) = \frac{1}{2}(\alpha_{i+1} + \alpha_{i+2})\iota_i(a), \\ \iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a \bullet b), \\ \iota_i(a) \circ \iota_i(b) = 2b(a, b)(e_{i+1} + e_{i+2}), \end{cases} \quad (2.4)$$

for any $\alpha_i, \beta_i \in k$, $a, b \in S$, $i = 0, 1, 2$, and where indices are taken modulo 3.

In [CE07b] it is shown that the Lie superalgebra of derivations of J is naturally isomorphic to the Lie superalgebra $\mathfrak{g}(S_1, S)$ in the first row of the Supermagic Square.

This is well-known for algebras, as $\mathfrak{g}(S_1, S)$ is isomorphic to the Lie algebra $\mathcal{T}(k, H_3(C))$ obtained by means of Tits construction (see [Eld04] and [BS03]), and this latter algebra is, by its own construction, the derivation algebra of $H_3(C)$. What was done in [CE07b, Section 3] is to make explicit this isomorphism $\mathfrak{g}(S_1, S) \cong \mathfrak{der} J$ and extend it to superalgebras.

To begin with, (2.4) shows that J is graded over $\mathbb{Z}_2 \times \mathbb{Z}_2$ with:

$$J_{(0,0)} = k^3, \quad J_{(1,0)} = \iota_0(S), \quad J_{(0,1)} = \iota_1(S), \quad J_{(1,1)} = \iota_2(S)$$

and, therefore, $\mathfrak{der} J$ is accordingly graded over $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$(\mathfrak{der} J)_{(i,j)} = \{d \in \mathfrak{der} J : d(J_{(r,s)}) \subseteq J_{(i+r, j+s)} \quad \forall r, s = 0, 1\}.$$

Moreover, the zero component is ([CE07b, Lemmas 3.4 and 3.5]):

$$(\mathfrak{der} J)_{(0,0)} = \{d \in \mathfrak{der} J : d(e_i) = 0 \quad \forall i = 0, 1, 2\},$$

and the linear map given by

$$\begin{aligned} \text{tri}(S) &\longrightarrow (\mathfrak{der} J)_{(0,0)} \\ (d_0, d_1, d_2) &\mapsto D_{(d_0, d_1, d_2)}, \end{aligned}$$

where

$$\begin{cases} D_{(d_0, d_1, d_2)}(e_i) = 0, \\ D_{(d_0, d_1, d_2)}(\iota_i(a)) = \iota_i(d_i(a)) \end{cases} \quad (2.5)$$

for any $i = 0, 1, 2$ and $a \in S$, is an isomorphism.

Given any two elements x, y in a Jordan superalgebra, the commutator (in the graded sense) of the left multiplications by x and y :

$$d_{x,y} = [L_x, L_y] \quad (2.6)$$

is a derivation. These derivations are called inner derivations. For any $i = 0, 1, 2$ and $a \in S$, consider the following inner derivation of the Jordan superalgebra J :

$$D_i(a) = 2d_{\iota_i(a), e_{i+1}} = 2[L_{\iota_i(a)}, L_{e_{i+1}}] \quad (2.7)$$

(indices modulo 3), where L_x denotes the multiplication by x in J . Note that the restriction of L_{e_i} to $\iota_{i+1}(S) \oplus \iota_{i+2}(S)$ is half the identity, so the inner derivation $[L_{\iota_i(a)}, L_{e_i}]$ is trivial on $\iota_{i+1}(S) \oplus \iota_{i+2}(S)$, which generates J . Hence

$$[L_{\iota_i(a)}, L_{e_i}] = 0 \quad (2.8)$$

for any $i = 0, 1, 2$ and $a \in S$. Also, $L_{e_0+e_1+e_2}$ is the identity map, so the bracket $[L_{\iota_i(a)}, L_{e_0+e_1+e_2}]$ is 0 and hence

$$D_i(a) = 2[L_{\iota_i(a)}, L_{e_{i+1}}] = -2[L_{\iota_i(a)}, L_{e_{i+2}}]. \quad (2.9)$$

A straightforward computation with (2.4) gives

$$\begin{aligned} D_i(a)(e_i) &= 0, \quad D_i(a)(e_{i+1}) = \frac{1}{2}\iota_i(a), \quad D_i(a)(e_{i+2}) = -\frac{1}{2}\iota_i(a), \\ D_i(a)(\iota_{i+1}(b)) &= -\iota_{i+2}(a \bullet b), \\ D_i(a)(\iota_{i+2}(b)) &= (-1)^{|a||b|}\iota_{i+1}(b \bullet a), \\ D_i(a)(\iota_i(b)) &= 2b(a, b)(-e_{i+1} + e_{i+2}), \end{aligned} \quad (2.10)$$

for any $i = 0, 1, 2$ and any homogeneous elements $a, b \in S$.

Denote by $D_i(S)$ the linear span of the $D_i(a)$'s, $a \in S$. Then the remaining components of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of $\mathfrak{der} J$ are given by ([CE07b, Lemma 3.11]):

$$(\mathfrak{der} J)_{(1,0)} = D_0(S), \quad (\mathfrak{der} J)_{(0,1)} = D_1(S), \quad (\mathfrak{der} J)_{(1,1)} = D_2(S).$$

Therefore, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of $\mathfrak{der} J$ becomes

$$\mathfrak{der} J = D_{\mathfrak{tri}(S)} \oplus \left(\bigoplus_{i=0}^2 D_i(S) \right) \quad (2.11)$$

On the other hand, $S_1 = k1$, with $1 \bullet 1 = 1$ and $b(1, 1) = 2$, so $\mathfrak{tri}(S_1) = 0$ and for the para-Hurwitz superalgebra S :

$$\begin{aligned} \mathfrak{g}(S_1, S) &= \mathfrak{tri}(S) \oplus \left(\bigoplus_{i=0}^2 \iota_i(S_1 \otimes S) \right) \\ &= \mathfrak{tri}(S) \oplus \left(\bigoplus_{i=0}^2 \iota_i(1 \otimes S) \right). \end{aligned}$$

Theorem 2.12. (See [CE07b, Theorem 3.13]) *Let S be a para-Hurwitz superalgebra over k and let J be the Jordan superalgebra of 3×3 hermitian matrices over the associated Hurwitz superalgebra. Then the linear map:*

$$\Phi : \mathfrak{g}(S_1, S) \longrightarrow \mathfrak{der} J, \quad (2.13)$$

such that

$$\begin{aligned} \Phi((d_0, d_1, d_2)) &= D_{(d_0, d_1, d_2)}, \\ \Phi(\iota_i(1 \otimes a)) &= D_i(a), \end{aligned}$$

for any $i = 0, 1, 2$, $a \in S$ and $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$, is an isomorphism of Lie superalgebras.

The Lie superalgebra $d_{J,J} = [L_J, L_J]$ is the Lie superalgebra $\mathfrak{inder} J$ of inner derivations of J . It turns out that $(\mathfrak{der} J)_{(r,s)} = (\mathfrak{inder} J)_{(r,s)}$ for $(r, s) \neq (0, 0)$, while

$$(\mathfrak{inder} J)_{(0,0)} = \sum_{i=0}^2 [L_{\iota_i(S)}, L_{\iota_i(S)}] = D_{\sum_{i=0}^2 \theta^i(t_{S,S})} = \Phi\left(\sum_{i=0}^2 \theta^i(t_{S,S})\right)$$

(recall that $\theta((d_0, d_1, d_2)) = (d_2, d_0, d_1)$ for any $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$).

In characteristic 3, $\mathfrak{tri}(S) = \sum_{i=0}^2 \theta^i(t_{S,S})$ if $\dim S = 1, 4$ or 8 ([Eld04]), and the same happens with $\mathfrak{tri}(S_{1,2})$ and $\mathfrak{tri}(S_{4,2})$, because of [CE07a, Corollaries 2.12 and 2.23], while for $\dim S = 2$, $\mathfrak{tri}(S)$ has dimension 2 and $\sum_{i=0}^2 \theta^i(t_{S,S}) = t_{S,S}$ has dimension 1 (see [Eld04]). In characteristic $\neq 3$, $\mathfrak{tri}(S) = \sum_{i=0}^2 \theta^i(t_{S,S})$ always holds.

Corollary 2.14. (See [CE07b, Corollary 3.15]) *Let S be a para-Hurwitz (super)algebra over k , and let J be the Jordan (super)algebra of 3×3 hermitian matrices over the associated Hurwitz (super)algebra. Then $\mathfrak{der} J$ is a simple Lie (super)algebra that coincides with $\mathfrak{inder} J$ unless the characteristic is 3 and $\dim S = 2$. In this latter case $\mathfrak{inder} J$ coincides with $[\mathfrak{der} J, \mathfrak{der} J]$, which is a codimension 1 simple ideal of $\mathfrak{der} J$.*

3. THE SECOND AND THIRD ROWS IN THE SUPERMAGIC SQUARE

The second row of the Supermagic Square is formed by the Lie superalgebras $\mathfrak{g}(S, S')$, where S and S' are symmetric composition superalgebras with $\dim S = 2$. Let S_2 be the split two dimensional para-Hurwitz algebra. Thus, S_2 is the para-Hurwitz algebra attached to the unital composition algebra $K = k \times k$, whose standard involution is given by $\overline{(\alpha, \beta)} = (\beta, \alpha)$ for any $\alpha, \beta \in k$. Then with $1 = (1, 1)$ and $u = (1, -1)$, the multiplication and norm in S_2 are given by:

$$\begin{aligned} 1 \bullet 1 &= 1, & 1 \bullet u &= u \bullet 1 = -u, & u \bullet u &= 1, \\ q(1) &= 1, & b(1, u) &= b(u, 1) = 0, & q(u) &= -1. \end{aligned}$$

Moreover, the triality Lie algebra of S_2 is (see [Eld04, Corollary 3.4]) the Lie algebra:

$$\mathfrak{tri}(S_2) = \{(\alpha_0 \sigma, \alpha_1 \sigma, \alpha_2 \sigma) : \alpha_0, \alpha_1, \alpha_2 \in k \text{ and } \alpha_0 + \alpha_1 + \alpha_2 = 0\},$$

where the linear map

$$\sigma = \frac{1}{2}\sigma_{1,u} \quad (3.1)$$

(recall (1.8)) satisfies $\sigma(1) = u$ and $\sigma(u) = 1$. Note that the element $t_{1,u} \in \mathbf{tri}(S_2)$ defined in (1.7) is given by:

$$t_{1,u} = (2\sigma, -\sigma, -\sigma). \quad (3.2)$$

Let now J be the Jordan superalgebra considered in equation (2.2). Its *structure Lie superalgebra* (or Lie multiplication superalgebra) $\mathbf{str}J$ is the subalgebra of the general Lie superalgebra $\mathbf{gl}(J)$ spanned by the Lie superalgebra of derivations $\mathbf{der} J$ of J and by the space L_J of left multiplications by elements in J . Since $[L_J, L_J]$ is a subalgebra of $\mathbf{der} J$, it follows that

$$\mathbf{str}J = \mathbf{der} J \oplus L_J.$$

Then the center of this Lie superalgebra is spanned by L_1 (the identity map). We will consider too the projective structure Lie superalgebra $\mathbf{pstr}J$, which is defined as the quotient of $\mathbf{str}J$ modulo its center:

$$\mathbf{pstr}J = \mathbf{str}J/kL_1 = \mathbf{str}J/kI,$$

where I denotes the identity map on J .

Then with the notations introduced in Section 2, we have:

Theorem 3.3. *Let S be a para-Hurwitz superalgebra over k and let J be the Jordan superalgebra of 3×3 hermitian matrices over the associated Hurwitz superalgebra. Then the isomorphism Φ in equation (2.13) extends to the following isomorphism of Lie superalgebras*

$$\Phi_2 : \mathbf{g}(S_2, S) \longrightarrow \mathbf{pstr}J \quad (3.4)$$

where:

- for any $(d_0, d_1, d_2) \in \mathbf{tri}(S)$, $\Phi_2((d_0, d_1, d_2)) = D_{(d_0, d_1, d_2)} + kI$,
- for any $\alpha_0, \alpha_1, \alpha_2 \in k$ with $\alpha_0 + \alpha_1 + \alpha_2 = 0$, the image under Φ of the element $(\alpha_0\sigma, \alpha_1\sigma, \alpha_2\sigma) \in \mathbf{tri}(S_2)$ is $L_{\alpha_2e_1 - \alpha_1e_2} + kI (= L_{\alpha_1e_0 - \alpha_0e_1} + kI = L_{\alpha_0e_2 - \alpha_2e_1} + kI)$,
- and for any $i = 0, 1, 2$, $a \in S$, $\Phi_2(\iota_i((\alpha 1 + \beta u) \otimes a)) = (\alpha D_i(a) + \beta L_{\iota_i(a)}) + kI$.

Proof. The proof is obtained by straightforward computations using the results in Section 2. Thus, for instance, for any $\alpha_i \in k$ ($i = 0, 1, 2$) with $\alpha_0 + \alpha_1 + \alpha_2 = 0$, and any $\alpha, \beta \in k$ and $a \in S$, we get:

$$[(\alpha_0\sigma, \alpha_1\sigma, \alpha_2\sigma), \iota_0((\alpha 1 + \beta u) \otimes a)] = \alpha_0\iota_0((\beta 1 + \alpha u) \otimes a),$$

which maps under Φ_2 to

$$\alpha_0(\beta D_0(a) + \alpha L_{\iota_0(a)}) + kI.$$

On the other hand, we have:

$$\begin{aligned} & [\Phi_2((\alpha_0\sigma, \alpha_1\sigma, \alpha_2\sigma)), \Phi_2(\iota_0((\alpha 1 + \beta u) \otimes a))] \\ &= [L_{\alpha_2e_1 - \alpha_1e_2}, \alpha D_0(a) + \beta L_{\iota_0(a)}] + kI \\ &= (-\alpha[D_0(a), L_{\alpha_1e_0 - \alpha_0e_1}] - \beta[L_{\iota_0(a)}, L_{\alpha_1e_0 - \alpha_0e_1}]) + kI \\ &= \alpha_0(\alpha L_{D_0(a)(e_1)} + \beta D_0(a)) + kI \\ &= \alpha_0(\beta D_0(a) + \alpha L_{\iota_0(a)}) + kI, \end{aligned}$$

where equations (2.7), (2.8), (2.9) and (2.10) have been used.

In a similar vein, for any homogeneous $a, b \in S$, the element

$$[\iota_0(1 \otimes a), \iota_0(u \otimes b)] = b(a, b)t_{1,u} = b(a, b)(2\sigma, -\sigma, \sigma)$$

(recall equations (3.1) and (3.2)) maps under Φ_2 to $b(a, b)L_{e_2-e_1} + kI$, while we have

$$\begin{aligned} & [\Phi_2(\iota_0(1 \otimes a)), \Phi_2(\iota_0(u \otimes b))] \\ &= [D_{\iota_0(a)}, L_{\iota_0(b)}] + kI \\ &= L_{D_0(a)(\iota_0(b))} + kI = b(a, b)L_{-e_1+e_2} + kI, \end{aligned}$$

as required.

The remaining computations needed to prove that Φ_2 is an isomorphism are similar to the ones above, and will be omitted. \square

Corollary 3.5. *The Lie superalgebras $\mathfrak{g}(S_2, S_{1,2})$ and $\mathfrak{g}(S_2, S_{4,2})$ in the Supermagic Square in characteristic 3 are isomorphic, respectively, to the projective structure Lie superalgebras of the Jordan superalgebras of hermitian 3×3 matrices over the unital composition superalgebras $B(1, 2)$ and $B(4, 2)$.*

Let us turn now our attention to the third row of the Supermagic Square.

Thus, let Q be a quaternion algebra (that is, a four dimensional unital composition algebra) over k , with multiplication denoted by juxtaposition, and denote by \bar{Q} the para-Hurwitz algebra with multiplication given by $x \bullet y = \bar{x}y = \bar{y}x$. Note that \bar{Q} is the para-Hurwitz algebra attached to the opposite algebra of Q (which is isomorphic to Q). In case Q is split, then it is isomorphic to the algebra of 2×2 matrices $\text{Mat}_2(k)$.

According to [Eld04, Corollary 3.4], the triality Lie algebra of \bar{Q} splits as:

$$\text{tri}(\bar{Q}) = \ker \pi_0 \oplus \ker \pi_1 \oplus \ker \pi_2, \quad (3.6)$$

where $\pi_i : \text{tri}(\bar{Q}) \rightarrow \mathfrak{so}(Q)$ is the projection onto the i th component. Moreover, let Q^0 denote the subspace of zero trace elements in Q , that is, the subspace orthogonal to the unity element. Then ([Eld04, Corollary 3.4]) we have:

$$\ker \pi_0 = \{(0, l_a \tau, -r_a \tau) : a \in Q^0\},$$

where l_a and r_a denote the left and right multiplications in \bar{Q} and $\tau : x \mapsto \bar{x}$ is the standard involution of Q . Therefore, for any $a \in Q^0$ and $x \in Q$, since $\bar{a} = \tau(a) = -a$, we get:

$$\begin{aligned} l_a \tau(x) &= a \bullet \bar{x} = \overline{a\bar{x}} = x\bar{a} = -xa = -R_a(x), \\ r_a \tau(x) &= \bar{x} \bullet a = \overline{\bar{x}a} = \bar{a}x = -ax = -L_a(x), \end{aligned}$$

where L_a and R_a denote the left and right multiplications by a in Q . Hence the ideal $\ker \pi_0$ above becomes:

$$\ker \pi_0 = \{(0, -R_a, L_a) : a \in Q^0\}.$$

and, similarly, $\ker \pi_1 = \{(L_a, 0, -R_a) : a \in Q^0\}$ and $\ker \pi_2 = \{(-R_a, L_a, 0) : a \in Q^0\}$.

Now, for any $a, b, x \in Q$, $q(x) = x\bar{x} = \bar{x}x$, so $b(a, b) = a\bar{b} + b\bar{a} = \bar{a}b + b\bar{a}$ and hence we have:

$$\begin{aligned} r_a l_b(x) &= \overline{\bar{b}xa} = \overline{\bar{x}ba} = \bar{a}bx = L_{\bar{a}b}(x), \\ l_a r_b(x) &= \overline{a\bar{x}b} = \overline{a\bar{b}x} = xb\bar{a} = R_{b\bar{a}}(x), \\ \sigma_{a,b}(x) &= b(a, x)b - b(b, x)a = (a\bar{x} + x\bar{a})b - a(\bar{b}x + x\bar{b}) = (-L_{\bar{a}b} + R_{\bar{a}b})(x), \\ \sigma_{a,b}(x) &= b(a, x)b - b(b, x)a = b(\bar{a}x + x\bar{a}) - (b\bar{x} + x\bar{b})a = (L_{b\bar{a}} - R_{b\bar{a}})(x), \\ \frac{1}{2}b(a, b)x - r_a l_b(x) &= \frac{1}{2}(\bar{a}b + \bar{b}a)x - L_{\bar{a}b}(x) = \frac{1}{2}L_{\bar{b}a - \bar{a}b}(x), \\ \frac{1}{2}b(a, b)x - l_a r_b(x) &= \frac{1}{2}x(\bar{a}b + \bar{b}a) - R_{b\bar{a}}(x) = \frac{1}{2}R_{a\bar{b} - b\bar{a}}(x). \end{aligned}$$

Therefore, the element $t_{a,b} \in \mathfrak{tri}(\bar{Q})$ in (1.7) becomes:

$$\begin{aligned} t_{a,b} &= (\sigma_{a,b}, \frac{1}{2}b(a,b)1 - r_a l_b, \frac{1}{2}b(a,b) - l_a r_b) \\ &= \frac{1}{2}(-L_{a\bar{b}-b\bar{a}} + R_{\bar{a}b-\bar{b}a}, -L_{\bar{a}b-\bar{b}a}, R_{a\bar{b}-b\bar{a}}). \end{aligned}$$

It must be noticed that the subspace Q^0 of trace zero elements is a three dimensional simple Lie algebra under the commutator, and that any three dimensional simple Lie algebra appears in this way. Moreover, given any Jordan algebra H , Tits considered in [Tit62] the Lie algebra defined on the vector space

$$\mathcal{T}(Q, H) = (Q^0 \otimes H) \oplus \mathfrak{der} H \quad (3.7)$$

endowed with the bracket given by:

- the restriction to $\mathfrak{der} H$ is the commutator in $\mathfrak{der} H$,
- $[d, a \otimes x] = a \otimes d(x)$,
- $[a \otimes x, b \otimes y] = ([a, b] \otimes xy) - 2b(a, b)d_{x,y}$,

for any $a, b \in Q^0$, $x, y \in H$, and $d \in \mathfrak{der} H$. (Recall the definition of $d_{x,y}$ in (2.6).)

In the split case $Q = \text{Mat}_2(k)$, the resulting Lie algebra is the well-known Tits-Kantor-Koecher Lie algebra $TKK(H)$ of the Jordan algebra H . Besides, all the arguments involved work in the super setting, and thus H can be taken to be any Jordan superalgebra.

Let us return our attention to the Jordan superalgebra J of hermitian 3×3 matrices over a unital composition superalgebra C , with associated para-Hurwitz superalgebra denoted by S , as in equation (2.2). Consider the Lie superalgebra $\mathfrak{g}(\bar{Q}, S)$ in equation (1.6). As a vector space, \bar{Q} splits into the direct sum $\bar{Q} = k1 \oplus Q^0$, and this gives the following decomposition of $\mathfrak{g}(\bar{Q}, S)$:

$$\mathfrak{g}(\bar{Q}, S) = \mathfrak{g}(S_1, S) \oplus \mathfrak{tri}(\bar{Q}) \oplus (\oplus_{i=0}^2 \iota_i(Q^0 \otimes S))$$

Also recall the decomposition in equation (3.6).

Then, as in Theorem 3.3, the isomorphism Φ in equation (2.13) can be extended to $\mathfrak{g}(\bar{Q}, S)$:

Theorem 3.8. *Let S be a para-Hurwitz superalgebra over k and let J be the Jordan superalgebra of 3×3 hermitian matrices over the associated Hurwitz superalgebra. Then the isomorphism Φ in equation (2.13) extends to the following isomorphism of Lie superalgebras*

$$\Phi_3 : \mathfrak{g}(\bar{Q}, S) \longrightarrow \mathcal{T}(Q, J) = (Q^0 \otimes J) \oplus \mathfrak{der} J, \quad (3.9)$$

where:

- the restriction of Φ_3 to $\mathfrak{g}(S_1, S)$ coincides with Φ in Theorem 2.12,
- for any $a \in Q^0$, the elements $(0, -R_a, L_a) \in \ker \pi_0$, $(L_a, 0, -R_a) \in \ker \pi_1$ and $(-R_a, L_a, 0) \in \ker \pi_2$ map, respectively, to $\frac{1}{2}(a \otimes e_0)$, $\frac{1}{2}(a \otimes e_1)$, and $\frac{1}{2}(a \otimes e_2)$, (recall that equation (3.6) shows that $\mathfrak{tri}(\bar{Q}) = \ker \pi_0 \oplus \ker \pi_1 \oplus \ker \pi_2$),
- for any $i = 0, 1, 2$, $a \in Q^0$ and $x \in S$, $\Phi_3(\iota_i(a \otimes x)) = -\frac{1}{2}a \otimes \iota_i(x)$.

The proof is obtained by straightforward computations and thus will be omitted.

Since $\mathcal{T}(Q, J)$ is the Tits-Kantor-Koecher Lie superalgebra of the Jordan superalgebra J in case Q is the split quaternion algebra, the next corollary follows at once:

Corollary 3.10. *The Lie superalgebras $\mathfrak{g}(S_4, S_{1,2})$ and $\mathfrak{g}(S_4, S_{4,2})$ in the Supermagic Square in characteristic 3 are isomorphic, respectively, to the Tits-Kantor-Koecher Lie superalgebras of the Jordan superalgebras of hermitian 3×3 matrices over the unital composition superalgebras $B(1,2)$ and $B(4,2)$.*

4. THE LIE SUPERALGEBRA $\mathfrak{g}(S_{1,2}, S_{1,2})$

The tiny *Kaplansky superalgebra* ([Kap75, McC94]) is the three dimensional Jordan superalgebra $K_3 = K_{\bar{0}} \oplus K_{\bar{1}}$, with $K_{\bar{0}} = ke$ and $K_{\bar{1}} = kx + ky$, and with multiplication given by:

$$\begin{aligned} e^2 &= e, & ex &= \frac{1}{2}x = xe, & ey &= \frac{1}{2}y = ye, \\ xy &= e = -yx, & x^2 &= 0 = y^2. \end{aligned}$$

On the other hand, the simple ten dimensional Kac superalgebra K_{10} was originally constructed in [Kac77b] over algebraically closed fields of characteristic 0 by Lie-theoretical methods from a 3-grading of the exceptional Lie superalgebra $F(4)$. In characteristic 3, K_{10} is no longer simple but possesses a simple ideal K_9 of dimension 9. Shestakov ([She96] unpublished) noticed that K_9 is isomorphic to the tensor product (as superalgebras) $K_3 \otimes K_3$. Later on, it was proven in [BE02] that K_{10} appears as a direct sum $k1 \oplus (K_3 \otimes K_3)$, with a natural multiplication, in any characteristic, and in particular, if the characteristic is 3, then $K_9 = K_3 \otimes K_3$.

Assume in the remaining of this section that the characteristic of our ground field k is 3.

Take the unital composition superalgebra $B(1,2)$ in (1.2) and its para-Hurwitz counterpart $S_{1,2}$. Then take a symplectic basis $\{u, v\}$ of $(S_{1,2})_{\bar{1}} = V$, so that $\{1, u, v\}$ is a basis of $S_{1,2}$. Since the characteristic is 3, the multiplication of $S_{1,2}$ is given by (see (1.5)):

$$\begin{aligned} 1 \bullet 1 &= \bar{1}\bar{1} = 1, & 1 \bullet x &= \bar{1}\bar{x} = 1(-x) = -x = \frac{1}{2}x = x \bullet 1 \quad \text{for any } x \in (S_{1,2})_{\bar{1}}, \\ u \bullet v &= \bar{u}\bar{v} = (-u)(-v) = 1, & u \bullet u &= 0 = v \bullet v, \end{aligned}$$

and therefore $S_{1,2}$ is just the tiny Kaplansky superalgebra K_3 .

Hence, Kac superalgebra K_9 can be identified with the superalgebra $S_{1,2} \otimes S_{1,2}$.

Now, the triality Lie superalgebra of $S_{1,2}$ is computed in [EO02, Theorem 5.6] (see also [CE07a, Corollary 2.12]):

$$\mathfrak{tri}(S_{1,2}) = \{(d, d, d) : d \in \mathfrak{osp}(S_{1,2}, b)\}. \quad (4.1)$$

That is, $\mathfrak{tri}(S_{1,2})$ is isomorphic to the orthosymplectic Lie superalgebra on the three dimensional vector superspace $S_{1,2}$ relative to the polar form of its norm. Also, in [EO02, Theorem 5.8] it is proven that the Lie superalgebra of derivations of $S_{1,2}$ is the whole orthosymplectic Lie superalgebra $\mathfrak{osp}(S_{1,2}, b)$.

In [BE02, Theorem 2.8] it is proven that the Lie superalgebra of derivations of the Kac superalgebra $K_9 = K_3 \otimes K_3$ is the direct sum of the Lie superalgebras of derivations of the two copies of the tiny Kaplansky superalgebra involved. That is, we have:

$$\mathfrak{der} K_9 = (\mathfrak{der} K_3 \otimes I) \oplus (I \otimes \mathfrak{der} K_3),$$

where I denotes the identity map and given any homogeneous linear maps $\varphi, \psi \in \mathfrak{gl}(K_3)$, $\varphi \otimes \psi$ is the linear endomorphism in $\mathfrak{gl}(K_9) = \mathfrak{gl}(K_3 \otimes K_3)$ given by:

$$(\varphi \otimes \psi)(x \otimes y) = (-1)^{\psi x}(\varphi(x) \otimes \psi(y))$$

for any homogeneous elements $x, y \in K_3$.

Consider now the quaternion algebra Q over k with a basis $\{1, e_0, e_1, e_2\}$, where 1 is the unity element, and with

$$e_i^2 = -1, \quad e_i e_{i+1} = -e_{i+2} = -e_{i+1} e_i,$$

(indices modulo 3). Its norm is the regular quadratic form q with $q(1) = 1 = q(e_i)$, for $i = 0, 1, 2$ and where the basis above is orthogonal. Since the characteristic is 3, $q(e_0 + e_1 + e_2) = 0$, so the norm of Q represents 0 and hence Q is the split quaternion algebra, that is, it is isomorphic to $\text{Mat}_2(k)$. (An explicit isomorphism can be easily constructed.)

To avoid confusion, let us denote by b_q the polar form of q . The subspace of zero trace elements is $Q^0 = ke_0 + ke_1 + ke_2$.

The Lie superalgebra $\mathcal{T}(Q, K_9)$ (see equation (3.7)), which is isomorphic to the Tits-Kantor-Koecher Lie superalgebra of K_9 as Q is split, is given by

$$\mathcal{T}(Q, J) = (Q^0 \otimes K_9) \oplus \mathfrak{der} K_9,$$

and hence it decomposes as:

$$\mathcal{T}(Q, K_9) = (\oplus_{i=0}^2 e_i \otimes (K_3 \otimes K_3)) \oplus ((\mathfrak{der} K_3 \otimes I) \oplus (I \otimes \mathfrak{der} K_3)),$$

that is, the direct sum of three copies of the tensor product $K_3 \otimes K_3$ (or $S_{1,2} \otimes S_{1,2}$) and two copies of $\mathfrak{der} K_3$, which is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}(S_{1,2}, b)$, exactly the situation that occurs for the Lie superalgebra $\mathfrak{g}(S_{1,2}, S_{1,2}) = \mathfrak{tri}(S_{1,2}) \oplus \mathfrak{tri}(S_{1,2}) \oplus (\oplus_{i=0}^2 \iota_i(S_{1,2} \otimes S_{1,2}))$.

Theorem 4.2. *Let k be a field of characteristic 3. Then the Lie superalgebra $\mathfrak{g}(S_{1,2}, S_{1,2})$ is isomorphic to the Tits-Kantor-Koecher Lie superalgebra of the Kac superalgebra K_9 .*

Proof. It has been checked above that both Lie superalgebras, $\mathfrak{g}(S_{1,2}, S_{1,2})$ and the Tits-Kantor-Koecher Lie superalgebra $\mathcal{T}(Q, K_9)$, split as vector superspaces into direct sums of isomorphic summands. Let us consider the explicit linear isomorphism:

$$\Psi : \mathfrak{g}(S_{1,2}, S_{1,2}) \longrightarrow \mathcal{T}(Q, K_9),$$

given by:

- $\Psi((d, d, d)) = d \otimes I \in \mathfrak{der} K_9$, for any (d, d, d) in the first copy of $\mathfrak{tri}(S_{1,2})$ in $\mathfrak{g}(S_{1,2}, S_{1,2})$,
- $\Psi((d', d', d')) = I \otimes d'$, for any (d', d', d') in the second copy of $\mathfrak{tri}(S_{1,2})$ in $\mathfrak{g}(S_{1,2}, S_{1,2})$,
- $\Psi(\iota_i(x \otimes x')) = e_i \otimes (x \otimes x')$, for any $i = 0, 1, 2$ and $x, x' \in S_{1,2}$,

and let us check that it is an isomorphism of Lie superalgebras.

In order to prove that Ψ is indeed a Lie superalgebra isomorphism, the only nontrivial point is to prove that

$$\Psi([\iota_i(x \otimes x'), \iota_i(y \otimes y')]) = [\Psi(\iota_i(x \otimes x')), \Psi(\iota_i(y \otimes y'))]$$

for any homogeneous, $x, x', y, y' \in S_{1,2} = K_3$ and $i = 0, 1, 2$. The symmetry of the constructions shows that it is enough to deal with $i = 0$. But the description of $\mathfrak{tri}(S_{1,2})$ in (4.1), together with equations (1.6), (1.7) and (1.8) give:

$$\begin{aligned} [\iota_0(x \otimes x'), \iota_0(y \otimes y')] &= (-1)^{xx' + xy' + yy'} b(x', y') t_{x,y} + (-1)^{x'y} b(x, y) t_{x',y'} \\ &= (-1)^{xx' + xy' + yy'} b(x', y') (\sigma_{x,y}, \sigma_{x,y}, \sigma_{x,y}) \\ &\quad + (-1)^{x'y} b(x, y) (\sigma_{x',y'}, \sigma_{x',y'}, \sigma_{x',y'}), \end{aligned}$$

which maps, under Ψ to:

$$\begin{aligned} \Psi([\iota_0(x \otimes x'), \iota_0(y \otimes y')]) \\ &= (-1)^{xx' + xy' + yy'} b(x', y')(\sigma_{x,y} \otimes I) + (-1)^{x'y} b(x, y)(I \otimes \sigma_{x',y'}), \\ &= (-1)^{x'y} ((\sigma_{x,y} \otimes b(x', y')I) + (b(x, y)I \otimes \sigma_{x',y'})) \in \mathfrak{der} K_9. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} [\Psi(\iota_0(x \otimes x')), \Psi(\iota_0(y \otimes y'))] \\ &= [e_0 \otimes (x \otimes x'), e_0 \otimes (y \otimes y')] \\ &= -2b_q(e_0, e_0)[L_{x \otimes x'}, L_{y \otimes y'}] \quad (\text{recall the product in (3.7)}) \\ &= -[L_{x \otimes x'}, L_{y \otimes y'}] \quad (\text{as } q(e_0) = 1, \text{ so } b_q(e_0, e_0) = 2 = -1) \\ &= -(-1)^{x'y} \frac{1}{2} (([L_x, L_y] \otimes b(x', y')I) + (b(x, y)I \otimes [L_{x'}, L_{y'}])) \\ &\quad (\text{by [BE02, (2.3)]}) \\ &= (-1)^{x'y} ((\sigma_{x,y} \otimes b(x', y')I) + (b(x, y)I \otimes \sigma_{x',y'})) \\ &\quad (\text{because of [BE02, (1.6)]}) \\ &= \Psi([\iota_0(x \otimes x'), \iota_0(y \otimes y')]), \end{aligned}$$

as required. \square

The Lie superalgebra $\mathfrak{g}(S_1, S_{1,2})$ is a subalgebra of $\mathfrak{g}(S_{1,2}, S_{1,2})$. The restriction of the isomorphism Ψ in the proof of Theorem 4.2 gives our last result:

Corollary 4.3. *Let k be a field of characteristic 3. Then the Lie superalgebra $\mathfrak{g}(S_1, S_{1,2})$ in the Supermagic Square is isomorphic to the Tits-Kantor-Koecher Lie superalgebra of the tiny Kaplansky superalgebra K_3 .*

Note that the Lie superalgebra $\mathfrak{g}(S_1, S_{1,2})$ is known to be isomorphic to the projective special Lie superalgebra $\mathfrak{psl}_{2,2}$ (see [CE07a]).

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